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# Analytical Solution for the American Options with Stochastic Volatility Using Barrier Options

# 隨機波動美式選擇權之解析解：障礙選擇權法

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# Analytical Solution for the American Options with Stochastic Volatility Using Barrier Options


## Abstract

This paper extends the work of Heston (1993) and integrates the Richardson extrapolation technique and Barrier options framework for developing analytical solution for stochastic volatility American options. By using large sample least-square Monte Carlo Simulations as the benchmarks, we prove that our model is accurate and efficient from the results of numerical experiments. Finally, we show that our stochastic volatility American option model is superior in pricing than the traditional constant volatility American option model from the empirical tests of Taiwan's put warrant market.



Keywords: analytical solution, American option, stochastic volatility, barrier option


## 1. Introduction

Unlike the European, American options have a higher value because holders are allowed to exercise them at any point of time until the expiration date. Pricing an American call option is easy because it has the same value as a European when no dividends are paid. However, pricing an American put option is extremely complex, stemming from the fact that the optimal-exercise policy on which the American option depends is unknown. Though numerical methods can be used for the pricing, the results obtained may not counteract the dynamic market change because they are time consuming and computationally more demanding. In contrast, analytical solutions are accurate and efficient formula forms which can intuitively describe phenomena such as monotone and convergence that cannot be proved by numerical methods. Hence, deriving an analytical solution for American put options is advantageous.

Much literature existing on pricing and hedging American options assumes volatility to be constant. However, we should reconsider such assumption since empirical studies have actually shown that the volatility in the real world is stochastic. Therefore, we assume that a more reasonable way of evaluating American put options is to relax the restriction of constant volatility.

The purpose of this paper is to derive an analytical solution for American put options with stochastic volatility by using barrier options. Barrier options are contingent claims whose value depends upon their behavior at various boundaries (Ingersoll (1998)). A down-and-out put option is one of the prototypical barrier options that will expire if the stock price ever falls below the knock-out barrier. Since American options are similar to barrier options with the exercise boundary treated as the barrier and the payoff as the rebate, we utilize such similarity to develop a stochastic volatility discrete down-and-out put option model under the Heston (1993) framework. The model is further modified into a stochastic volatility discrete American put option. Then, Richardson extrapolation is applied to obtain the analytical solution.

To appraise the pricing performance of our model, numerical analyses are conducted to verify the accuracy in comparison with large sample least-square Monte Carlo Simulations (LSM). The encouraging results show that our model is accurate in pricing. In addition, we observe the difference in option price between our stochastic volatility model and the traditional constant volatility model as the variance increases. The obvious difference lies between the two models has proved that it is relatively impractical to apply constant volatility in the past. Empirical analyses are also performed on Taiwan’s put warrants. It is evident that stochastic volatility model is indeed more realistic in the actual market. Therefore, our derived model is a practical tool for the option practitioners.

## 2. Literature Review

### 2.1 Stochastic Volatility

Hull and White (1987) are the first to suggest using the idea of stochastic volatility to relax the constant volatility assumption in Black-Scholes (1973) model. In Hull and White (1987) model, the stock price and the asset volatility follow their respective diffusion process. The greatest restriction is the zero correlation between both. The volatility of the model adopts the lognormal process and obtains the power series approximation of the European option. Since then, stochastic volatility has been widely paid attention to.

On the other hand, Stein and Stein (1991) adopt a different viewpoint. By obeying the mathematical Ornstein-Uhlenbeck process and using a separate numerical integration, a closed-form solution is derived. However, this model still fails to relax the unreasonable assumption of no correlation between the stock price and the volatility.

Of all, Heston (1993) model makes the most contribution. Heston (1993) describes the volatility of the underlying asset as a dynamic one by using square root model and derives a closed-form solution for European options. Heston (1993) model uses Fourier transforms technique and characteristic function to calculate the probability of in-the-money when the option reaches its expiration date. The stock price diffusion process of Heston (1993) is identical to Black-Scholes (1973) while the formula form of the closed-form is similar except for the volatility that changes with time. The allowance of correlation between the stock price and the volatility in Heston (1993) model is crucial because mutual influence is supposed to exist between these two variables theoretically. Therefore, we extend Heston (1993) model for the stochastic volatility calculation since it is more conformed to the real world status.

### 2.2 Analytical Solutions

Three approaches are adopted in the pricing of American options. The first approach is using the integral method to calculate under the risk neutral probability measure (Geske and Johnson (1984)). However, it can be computationally time consuming when one more exercisable is added for accuracy. The second approach is seeking solution directly from Black and Scholes (1973) partial differential equation (PDE) (Barone-Adesi and Whaley (1987)). Although BAW (1987) approximation is fast, the serious drawback of it is the lack of accuracy especially under long maturity options. The third approach is the use of down-and-out put option (Ingersoll (1998); Sbuelz (2004)). Barrier options not only determine the final stock price on the expiration date, but also consider whether the options hit the barrier before the expiry time. A down-and-out put option is valuable when the stock price falls below the strike price and does not hit the barrier. American options possess similar characteristic as barrier options. When both options hit a certain level, the value will begin to change. Hence, Ingersoll (1988) and Sbuelz (2004) add a rebate to the down-and-out put option model in order to satisfy the immediate exercise value when the stock price hits the barrier. Unfortunately, these models are derived under constant volatility. Therefore, we aim to modify the down-and-out put option in the derivation of the analytical solution for American options.

## 3. The Model

We intuitively think that a stochastic volatility down-and-out put option model can be derived by developing a down-and-out put option model under the Heston (1993) framework. However, difficulty arises because the accuracy of the down-and-out probability cannot be obtained correctly under Heston (1993) model; thus obstructing the subsequent development. Therefore, we used another alternative to circumvent the problem. With reference to Griebsch and Wystup (2008) in developing the model for fader option evaluation, it has directed us with a solution. Under the same Heston (1993) framework, Griebsch and Wystup (2008) used the characteristic function proposed by Shepard (1991) to calculate discrete probability distribution. This is equivalent to calculating a discrete down-and-in probability. If this probability is obtained, we can develop a discrete down-and-in put and eventually a stochastic volatility down-and-out put option model. When an immediate exercise value is added, the model will become a discrete stochastic volatility American put option. Then, we can apply the Richardson extrapolation technique to approach and obtain the analytical solution.

### 3.1 Heston Model

The diffusion process of the stock price and the variance follows the setting of Heston (1993) stochastic volatility model

$$dS = \mu S \; dt + \sqrt{\upsilon} S \; dz_1 \tag{1}$$

$$d\upsilon = \kappa\left(\theta - \upsilon\right) \; dt + \sigma\sqrt{\upsilon} \; dz_2 \tag{2}$$

where $\kappa$ is the rate of mean reversion, $\theta$ is the long term variance and $\sigma$ represents the volatility of variance. They are all constant parameters. $z_1$ and $z_2$ are two correlated Brownian motions. $\rho$ is the correlation coefficient of these two Brownian motions. In equation (1), the stock price $S$ is a geometric Brownian motion and its volatility changes with time. In equation (2), current variance $\upsilon$ follows the square root process (Cox-Ingersoll-Ross (1985) process).

Similar to Black-Scholes (1973) formula, Heston (1993) guessed the solution form of this European call with stochastic volatility

$$C\left(S, K, T\right) = SF_1 - Ke^{-rT} F_2 \tag{3}$$

$$F_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{\exp\left(-iu \ln K\right)\varphi_j\left(u\right)}{iu}\right] du \tag{4}$$

where $j = 1, 2$. $K$ is the strike price, $r$ represents the risk-free interest rate, $T$ denotes the time to maturity, $\mathrm{Re}\left[\cdot\right]$ stands for real part, function $\varphi_j\left(u\right)$ is the characteristic function **(**Griebsch and Wystup (2008)**)**. $F_j$ is the conditional probability that the option expires in-the-money. The form of stochastic volatility European put can be obtained through put-call parity.

$$P\left(S, K, T\right) = C\left(S, K, T\right) + Ke^{-rT} - S \tag{5}$$

### 3.2 Discrete down-and-out put option with stochastic volatility

In Griebsch and Wystup (2008) paper, the probability in the fader option is calculated by taking the probability of stock price under high barrier H minus the probability of stock price under low barrier L. By just considering the probability under low barrier L, it is equivalent to obtaining the discrete down-and-in probability. With it, a discrete stochastic volatility down-and-in put option can be developed. Below is the illustration of the discrete down-and-in probability calculation.

With the *n* dimensional characteristic function and the examples and principle provided by Shephard (1991), we are able to calculate this probability distribution on our own.

1-dimensional：

$$F_j(c_1)=\mathrm{Prob}_j\left(x_{t_1}\le c_1\right)=\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \frac{\varphi_j(u)e^{-ic_1u}}{iu}du \tag{6}$$

2-dimensional：

$$\begin{aligned}
&F_j(c_1,c_2)\\
&=\mathrm{Prob}_j\left(x_{t_1}\le c_1, x_{t_2}\le c_2\right)\\
&=\frac{1}{4}-\frac{1}{2\pi}\int_0^\infty \mathrm{Re}\left[\frac{\varphi_j(u_1,0)e^{-ic_1u_1}}{iu_1}\right]du_1-\frac{1}{2\pi}\int_0^\infty \mathrm{Re}\left[\frac{\varphi_j(0,u_2)e^{-ic_2u_2}}{iu_2}\right]du_2\\
&\quad-\frac{1}{2\pi^2}\iint_{R_+^2}\mathrm{Re}\left[\frac{\varphi_j(u_1,u_2)e^{-ic_1u_1-ic_2u_2}-\varphi_j(u_1,-u_2)e^{-ic_1u_1+ic_2u_2}}{u_1u_2}\right]du_1du_2
\end{aligned} \tag{7}$$

3-dimensional：

$$\begin{aligned}
&F_j(c_1,c_2,c_3)\\
&=\mathrm{Prob}_j\left(x_{t_1}\le c_1, x_{t_2}\le c_2, x_{t_3}\le c_3\right)\\
&=\frac{1}{8}-\frac{1}{4\pi}\int_0^\infty \mathrm{Re}\left[\frac{\varphi_j(u_1,0,0)e^{-ic_1u_1}}{iu_1}\right]du_1-\frac{1}{4\pi}\int_0^\infty \mathrm{Re}\left[\frac{\varphi_j(0,u_2,0)e^{-ic_2u_2}}{iu_2}\right]du_2\\
&\quad-\frac{1}{4\pi}\int_0^\infty \mathrm{Re}\left[\frac{\varphi_j(0,0,u_3)e^{-ic_3u_3}}{iu_3}\right]du_3\\
&\quad-\frac{1}{4\pi^2}\iint_{R_+^2}\mathrm{Re}\left[\frac{\varphi_j(u_1,u_2,0)e^{-ic_1u_1-ic_2u_2}-\varphi_j(u_1,-u_2,0)e^{-ic_1u_1+ic_2u_2}}{u_1u_2}\right]du_1du_2\\
&\quad-\frac{1}{4\pi^2}\iint_{R_+^2}\mathrm{Re}\left[\frac{\varphi_j(u_1,0,u_3)e^{-ic_1u_1-ic_3u_3}-\varphi_j(u_1,0,-u_3)e^{-ic_1u_1+ic_3u_3}}{u_1u_3}\right]du_1du_3\\
&\quad-\frac{1}{4\pi^2}\iint_{R_+^2}\mathrm{Re}\left[\frac{\varphi_j(0,u_2,u_3)e^{-ic_2u_2-ic_3u_3}-\varphi_j(0,u_2,-u_3)e^{-ic_2u_2+ic_3u_3}}{u_2u_3}\right]du_2du_3\\
&\quad+\frac{1}{4\pi^3}\iiint_{R_+^3}\mathrm{Im}\left[\begin{array}{l}\dfrac{\varphi_j(u_1,u_2,u_3)e^{-ic_1u_1-ic_2u_2-ic_3u_3}-\varphi_j(u_1,-u_2,u_3)e^{-ic_1u_1+ic_2u_2-ic_3u_3}}{u_1u_2u_3}\\ -\dfrac{\varphi_j(u_1,u_2,-u_3)e^{-ic_1u_1-ic_2u_2+ic_3u_3}-\varphi_j(u_1,-u_2,-u_3)e^{-ic_1u_1+ic_2u_2+ic_3u_3}}{u_1u_2u_3}\end{array}\right]du_1du_2du_3
\end{aligned} \tag{8}$$

for $j=1,2$ ($\mathrm{Re}[\cdot]$: real part, $\mathrm{Im}[\cdot]$: imaginary part). $\mathrm{Prob}_j(\cdot)$ is the probability under different measures.

Through the probability distribution, the discrete stochastic volatility down-and-in put option (refer to equations (10) and (19)) can be calculated with an extension of three discrete exercisable time points. With in-put parity, a discrete stochastic volatility down-and-out put option can hence be formed.

### 3.3 A discrete American put with stochastic volatility

To modify the discrete stochastic volatility down-and-out put model (refer to equations (11) and (20)) into a discrete stochastic volatility American put (Bermuda option), an immediate exercise value has to be added. For American put options, the time for optimal early exercise is when the stock price falls below or hits the critical price. The concept of the critical price is the same as the barrier. In barrier options, when the stock price hits the barrier, the value of the option will be zero. However, for American options, when the stock price hits the critical price, an immediate exercise value can be obtained. To avoid a null value, we add this immediate exercise value to the discrete stochastic volatility down-and-out put option model so that a discrete stochastic volatility American put (Bermuda option) can be formed.

#### 3.3.1 The Value of Immediate Exercise

For American put options, the immediate exercise value (refer to equations (12), (21) and (22)) is the difference between the strike price and the critical price. It can be calculated by taking the critical price as the strike price.

In Black-Scholes pricing formula $d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$, $N(d_2)$ represents the probability of in-the-money condition, and the stock price is more than the strike price on the expiration date. We intuitively think that by replacing the strike price $K$ with the critical price in $\ln(S/K)$, the obtained probability of $N(d_1)$ and $N(d_2)$ will become the probability of in-the-money calculated with the critical price instead of the strike price.

#### 3.3.2 The Critical Price of the Model

Critical price $\bar{S}$ is an important element when pricing stochastic volatility American put options because it represents the optimal exercise time. For American options, the optimal-exercise policy can be presented as the exercise boundary in price-time space. The boundary partitioned the space into a hold region and an exercise region. In a put option, early exercise occurs when the stock price falls below or hits the critical price, and there should be a holding when it is above the critical price. Therefore, critical price can be determined when the immediate exercise value is equivalent to the holding value. This can be illustrated in a simple put option below: $K - \bar{S} = P(\bar{S}, K, T)$ for some $S = \bar{S}$ and any $T$. An initial value of the critical price will be given first. This value also represents the initial stock price. Then, substitute the given value into the model and solve it iteratively with the bisection method until the immediate exercise value is equivalent to the holding value. The result obtained will hence be the critical price.

### 3.4 Analytical Solution for American Put with Stochastic Volatility

After obtaining a discrete stochastic volatility American put model (Bermuda option) and a calculated critical price, three-point Richardson extrapolation is applied to evaluate American put options with stochastic volatility.

Let $P_1$ be a pure stochastic volatility European put that can only be exercised at expiration time $T$ (Equivalent to equation (5)).

$$P_1 = P\left(S,K,T\right) \tag{9}$$

Let $P_{2-di}$ be a stochastic volatility down-and-in put. A barrier $H$ at time $T/2$. The option is valuable only if the stock price hits or falls below the barrier. When it comes to expiration time $T$, the option must be valuable.

$$\begin{aligned} P_{2-di} &= Ke^{-rT}\left[\text{Prob}_2\left(x_{T/2} \le H, x_T \le K\right)\right] - S\left[\text{Prob}_1\left(x_{T/2} \le H, x_T \le K\right)\right] \\ &= Ke^{-rT}\left[F_2\left(H,K\right)\right] - S\left[F_1\left(H,K\right)\right] \end{aligned} \tag{10}$$

$P_{2-do}$ is a stochastic volatility down-and-out put and can be obtained:

$$P_{2-do}\left(S,K,T,H\right) = P\left(S,K,T\right) - P_{2-di} \tag{11}$$

Let $P_{2-exercise}$ be a stochastic volatility European put. Expiration time be $T/2$. The critical price is taken as the strike price at $T/2$ so that the value of the put will be the immediate exercise value.

$$P_{2-exercise} = P\left(S, \bar{S}_{T/2}, T/2\right) \tag{12}$$

Let $P_2$ be a stochastic volatility American put with. Early exercise can only be determined at time $T/2$. ($P_2$ is equivalent to Bermuda option)

$$P_2\left(S,K,T,\bar{S}_{T/2}\right) = P_{2-do}\left(S,K,T,\bar{S}_{T/2}\right) + P_{2-exercise} \tag{13}$$

Since the maturity interval from $T/2$ to $T$ is only a pure European put (see Figure 1), critical price $\bar{S}_{T/2}$ at $T/2$ can be obtained by using bisection method to solve iteratively.

$$K - \bar{S}_{T/2} = P\left(\bar{S}_{T/2}, K, T/2\right) \tag{14}$$

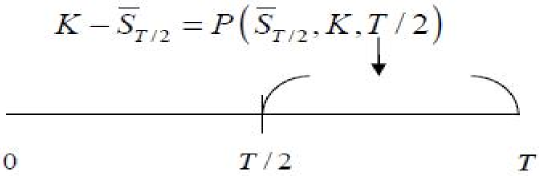


**Figure 1: The critical price $\overline{S}_{T/2}$ at $T/2$**

Let $P_{3-di}$ be a stochastic volatility down-and-in put. There is a barrier $H$ at $T/3$ and $2T/3$ respectively. The option is valuable only if the stock price hits or falls below the barrier. It has to be valuable when it comes to expiration time $T$ . However, this probability must be considered in detail in three cases:

1) The stock price hits or falls below the barrier at $T/3$. However, the stock price does not hit at $2T/3$ and is above the barrier. When it comes to expiration time $T$ , it has to be valuable.

2) The stock price does not hit at $T/3$ and is above the barrier. However, it hits or falls below the barrier at $2T/3$. When it comes to expiration time $T$ , it has to be valuable.

3) The stock price hits or falls below the barrier at both $T/3$ and $2T/3$. When it comes to expiration time $T$ , it has to be valuable.

The above three conditions are all down-and-in put probabilities, so we can use the concept of set:

$$\begin{aligned}&\mathrm{Prob}_j\left(x_{T/3}\le H,x_{2T/3}>H,x_T\le K\right)\\&=\mathrm{Prob}_j\left(x_{T/3}\le H,x_T\le K\right)-\mathrm{Prob}_j\left(x_{T/3}\le H,x_{2T/3}\le H,x_T\le K\right)\\&=F_j\left(H,K\right)-F_j\left(H,H,K\right)\end{aligned} \tag{15}$$

$$\begin{aligned}&\mathrm{Prob}_j\left(x_{T/3}>H,x_{2T/3}\le H,x_T\le K\right)\\&=\mathrm{Prob}_j\left(x_{2T/3}\le H,x_T\le K\right)-\mathrm{Prob}_j\left(x_{T/3}\le H,x_{2T/3}\le H,x_T\le K\right)\\&=F_j\left(H,K\right)-F_j\left(H,H,K\right)\end{aligned} \tag{16}$$

$$\mathrm{Prob}_j\left(x_{T/3}\le H,x_{2T/3}\le H,x_T\le K\right)=F_j\left(H,H,K\right) \tag{17}$$

Thus, the probability of a down-and-in ($\mathrm{Prob}_{j-di}$) is the sum of the three conditions：

$$\begin{aligned}\mathrm{Prob}_{j-di}&=\mathrm{Prob}_j\left(x_{T/3}\le H,x_T\le K\right)+\mathrm{Prob}_j\left(x_{2T/3}\le H,x_T\le K\right)\\&\quad-\mathrm{Prob}_j\left(x_{T/3}\le H,x_{2T/3}\le H,x_T\le K\right)\end{aligned} \tag{18}$$

for $j=1,2$

$P_{3-di}$ can be obtained：

$$P_{3-di} = Ke^{-rT}\left[\text{Prob}_{2-di}\right] - S\left[\text{Prob}_{1-di}\right] \tag{19}$$

$P_{3-do}$ is a stochastic volatility down-and-out put and can be obtained：

$$P_{3-do}\left(S,K,T,H,H\right) = P\left(S,K,T\right) - P_{3-di} \tag{20}$$

Let $P_{31-exercise}$ be a stochastic volatility European put. Expiration time be $T/3$. The critical price is taken as the strike price so that the value of the put will be the immediate exercise value.

$$P_{31-exercise} = P\left(S,\overline{S}_{T/3},T/3\right) \tag{21}$$

Let $P_{32-exercise}$ be a model of $P_{2-di}$. Expiration time be $2T/3$. The critical price at $T/3$ is taken as the barrier, and the critical price at $2T/3$ be the strike price. The value of the put is the same as the immediate exercise value.

$$\begin{aligned} & P_{32-exercise} \\ & = Ke^{-rT}\left[\text{Prob}_2\left(x_{T/3} \le \overline{S}_{T/3}, x_{2T/3} \le \overline{S}_{2T/3}\right)\right] - S\left[\text{Prob}_1\left(x_{T/3} \le \overline{S}_{T/3}, x_{2T/3} \le \overline{S}_{2T/3}\right)\right] \\ & = Ke^{-rT}\left[F_2\left(\overline{S}_{T/3},\overline{S}_{2T/3}\right)\right] - S\left[F_1\left(\overline{S}_{T/3},\overline{S}_{2T/3}\right)\right] \end{aligned} \tag{22}$$

Let $P_3$ be a stochastic volatility American put. Early exercise can only be determined at time $T/3$ and $2T/3$ . ($P_3$ is equivalent to Bermuda option)

$$P_3\left(S,K,T,\overline{S}_{T/3},\overline{S}_{2T/3}\right) = P_{3-do}\left(S,K,T,\overline{S}_{T/3},\overline{S}_{2T/3}\right) + P_{31-exercise} + P_{32-exercise} \tag{23}$$

Since the maturity interval from $2T/3$ to $T$ is only a pure European put (see Figure 2), critical price $\overline{S}_{2T/3}$ at $2T/3$ can be obtained by using the bisection method to solve iteratively.

$$K - \overline{S}_{2T/3} = P\left(\overline{S}_{2T/3},K,T/3\right) \tag{24}$$

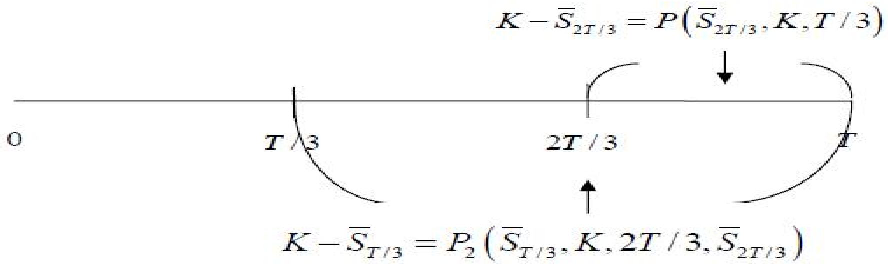


**Figure 2: The critical price $\overline{S}_{T/3}$ at $T/3$ and the critical price $\overline{S}_{2T/3}$ at $2T/3$**

The critical price $\bar{S}_{T/3}$ at $T/3$ requires $P_2$ model to solve because the maturity interval from $T/3$ to $T$ is a representation of a $P_2$ model itself. It can be exercised at either $2T/3$ or $T$. Hence, critical price $\bar{S}_{T/3}$ can still be obtained by using the bisection method to solve iteratively. (see Figure 3)

$$K-\bar{S}_{T/3}=P_2\left(\bar{S}_{T/3},K,2T/3,\bar{S}_{2T/3}\right) \tag{25}$$

Eventually, we derived $P_1$, $P_2$ and $P_3$. By applying the three-point Richardson extrapolation to approach (Geske and Johnson (1984)), we can obtain the analytical solution for the American put options with stochastic volatility:

$$P_{SV}=\frac{9}{2}P_3-4P_2+\frac{1}{2}P_1 \tag{26}$$

## 4. Numerical Analyses

Numerical analyses are carried out on our model for stochastic volatility American put options ($P_{SV}$). The programs are written in C++ and MATLAB. Since the least-square Monte Carlo Simulation (LSM) (Longstaff and Schwartz (2001)) model is intuitive, accurate, efficient and convenient to apply, we set the values calculated by it as the benchmark. Then, we compared the difference between stochastic volatility model and constant volatility model.

### 4.1 Benchmark and Setting of the Parameters

In the LSM method, we unified in adopting 100,000 paths, repeating 30 times and using the trading day as steps. We have tried setting both quadratic polynomial and cubic polynomial for basis function $f$. There is only a slight difference between the LSM value of both polynomials. However, the time consumed for cubic polynomial is much more than expected. Therefore, we considered until quadratic polynomial. The basis function $f$ used in LSM model is $f=a_0+a_1S+a_2S^2+a_3\upsilon+a_4\upsilon^2+a_5S\upsilon$ , $a_0,a_1,a_2,a_3,a_4,a_5$ are the parameters estimated by regression. The initial hypothesis of the parameters is set as: strike price $K$=50, risk-free interest rate r=5%, the rate of mean reversion $\kappa$=2, the long term variance $\theta$=30%, volatility of variance $\sigma$=22.5% and variance $\upsilon$=30%. The setting of the parameters follows the Heston (1993) paper. The correlation coefficient is divided into three conditions: $\rho$=0.5, $\rho$=0, $\rho$=-0.5. The stock price $S$ is set as: in-the-money (ITM) $S$ = 45, at-the-money (ATM) $S$ = 50 and out-the-money (OTM) $S$ =55.

We assume that if the relative error in price calculated by both $P_{SV}$ and LSM method is less than 1%, then $P_{SV}$ is of high accuracy. The relative error is calculated by first using the value of $P_{SV}$ minus the value of LSM method, and the difference obtained would then be divided by the value of LSM method.

$$\text{Relative error} = \frac{P_{SV} - P_{LSM}}{P_{LSM}} \tag{27}$$

In terms of calculating efficiency, we can use the time taken by the LSM method over the time taken by $P_{SV}$.

$$\text{Efficiency} = \frac{CPU\ time_{LSM}}{CPU\ time_{P_{SV}}} \tag{28}$$

### 4.2 The Accuracy of the Analytical Solution Model

Table 1 to Table 3 is the comparison between $P_{SV}$ and LSM method in ITM, ATM and OTM respectively. The parameters are the same as the initial setting. The option with the shortest time to maturity is one month, and the longest is three years. According to the tables, the relative errors shown are all less than 1%. This proves that our model is very accurate in different terms of options. Regardless of the value of correlation coefficient $\rho$, the value of the options increases with increasing time to maturity.

**Table 1 Comparison between $P_{SV}$ and LSM for in-the-money**

| | | $\rho$ =0.5 | | | $\rho$ =0 | | | $\rho$ =-0.5 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $S$ | $T$ (yr) | $P_{SV}$ | LSM (std) | relative error | $P_{SV}$ | LSM (std) | relative error | $P_{SV}$ | LSM (std) | relative error |
| 45 | 0.083 | 5.136 | 5.130 (0.008) | 0.109% | 5.117 | 5.109 (0.008) | 0.149% | 5.098 | 5.090 (0.006) | 0.146% |
| | 0.166 | 5.438 | 5.435 (0.010) | 0.044% | 5.397 | 5.390 (0.010) | 0.133% | 5.356 | 5.341 (0.009) | 0.277% |
| | 0.25 | 5.723 | 5.724 (0.013) | -0.013% | 5.669 | 5.659 (0.012) | 0.171% | 5.614 | 5.596 (0.013) | 0.324% |
| | 0.333 | 5.979 | 5.985 (0.012) | -0.091% | 5.916 | 5.910 (0.011) | 0.095% | 5.851 | 5.836 (0.012) | 0.262% |
| | 0.416 | 6.209 | 6.218 (0.017) | -0.140% | 6.139 | 6.136 (0.017) | 0.050% | 6.068 | 6.050 (0.014) | 0.305% |
| | 0.5 | 6.417 | 6.432 (0.020) | -0.226% | 6.343 | 6.344 (0.017) | -0.011% | 6.268 | 6.255 (0.015) | 0.203% |
| | 0.583 | 6.608 | 6.618 (0.017) | -0.146% | 6.531 | 6.531 (0.013) | 0.001% | 6.453 | 6.436 (0.014) | 0.265% |
| | 0.666 | 6.783 | 6.797 (0.019) | -0.204% | 6.705 | 6.707 (0.015) | -0.028% | 6.626 | 6.610 (0.017) | 0.243% |
| | 0.75 | 6.946 | 6.955 (0.017) | -0.128% | 6.868 | 6.861 (0.020) | 0.094% | 6.788 | 6.766 (0.021) | 0.331% |
| | 0.833 | 7.098 | 7.113 (0.019) | -0.213% | 7.020 | 7.022 (0.020) | -0.032% | 6.941 | 6.930 (0.022) | 0.163% |
| | 0.916 | 7.241 | 7.254 (0.016) | -0.176% | 7.164 | 7.159 (0.019) | 0.068% | 7.086 | 7.064 (0.020) | 0.316% |
| | 1 | 7.375 | 7.387 (0.018) | -0.161% | 7.300 | 7.297 (0.017) | 0.035% | 7.224 | 7.205 (0.019) | 0.263% |
| | 2 | 8.573 | 8.577 (0.020) | -0.052% | 8.536 | 8.520 (0.024) | 0.190% | 8.496 | 8.460 (0.025) | 0.427% |
| | 3 | 9.359 | 9.342 (0.023) | 0.174% | 9.367 | 9.321 (0.026) | 0.490% | 9.368 | 9.297 (0.025) | 0.763% |

Note: std is the standard deviation. The results are shown in percentage.

**Table 2 Comparison between $P_{SV}$ and LSM for in-the-money**

| | | $\rho$ =0.5 | | | $\rho$ =0 | | | $\rho$ =-0.5 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $S$ | $T$ (yr) | $P_{SV}$ | LSM std | relative error | $P_{SV}$ | LSM std | relative error | $P_{SV}$ | LSM std | relative error |
| 50 | 0.083 | 1.634 | 1.628 (0.006) | 0.346% | 1.634 | 1.628 (0.006) | 0.372% | 1.634 | 1.628 (0.006) | 0.365% |
| | 0.166 | 2.254 | 2.247 (0.009) | 0.310% | 2.254 | 2.247 (0.007) | 0.315% | 2.253 | 2.248 (0.006) | 0.218% |
| | 0.25 | 2.709 | 2.702 (0.009) | 0.256% | 2.708 | 2.699 (0.009) | 0.331% | 2.706 | 2.696 (0.009) | 0.378% |
| | 0.333 | 3.077 | 3.071 (0.010) | 0.186% | 3.076 | 3.070 (0.012) | 0.215% | 3.074 | 3.068 (0.011) | 0.194% |
| | 0.416 | 3.390 | 3.385 (0.013) | 0.139% | 3.390 | 3.387 (0.014) | 0.097% | 3.388 | 3.386 (0.015) | 0.059% |
| | 0.5 | 3.665 | 3.664 (0.015) | 0.020% | 3.666 | 3.664 (0.015) | 0.052% | 3.663 | 3.661 (0.015) | 0.055% |
| | 0.583 | 3.911 | 3.910 (0.011) | 0.010% | 3.912 | 3.908 (0.015) | 0.107% | 3.910 | 3.905 (0.016) | 0.123% |
| | 0.666 | 4.133 | 4.136 (0.012) | -0.053% | 4.136 | 4.137 (0.013) | -0.024% | 4.135 | 4.136 (0.014) | -0.043% |
| | 0.75 | 4.338 | 4.336 (0.014) | 0.037% | 4.342 | 4.336 (0.019) | 0.130% | 4.341 | 4.338 (0.018) | 0.067% |
| | 0.83 | 4.526 | 4.529 (0.017) | -0.057% | 4.532 | 4.535 (0.019) | -0.058% | 4.532 | 4.539 (0.018) | -0.156% |
| | 0.916 | 4.702 | 4.704 (0.013) | -0.047% | 4.710 | 4.709 (0.015) | 0.025% | 4.711 | 4.713 (0.016) | -0.050% |
| | 1 | 4.866 | 4.871 (0.015) | -0.100% | 4.876 | 4.879 (0.017) | -0.048% | 4.879 | 4.881 (0.020) | -0.044% |
| | 2 | 6.298 | 6.318 (0.024) | -0.311% | 6.341 | 6.354 (0.023) | -0.202% | 6.372 | 6.384 (0.023) | -0.186% |
| | 3 | 7.213 | 7.230 (0.022) | -0.230% | 7.294 | 7.303 (0.024) | -0.132% | 7.359 | 7.353 (0.023) | 0.080% |

Note: std is the standard deviation. The results are shown in percentage.

**Table 3 Comparison between $P_{SV}$ and LSM for out-the-money**

| | | $\rho$ =0.5 | | | $\rho$ =0 | | | $\rho$ =-0.5 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $S$ | $T$ (yr) | $P_{SV}$ | LSM std | relative error | $P_{SV}$ | LSM std | relative error | $P_{SV}$ | LSM std | relative error |
| 55 | 0.083 | 0.261 | 0.260 (0.003) | 0.415% | 0.285 | 0.284 (0.003) | 0.212% | 0.307 | 0.308 (0.003) | -0.428% |
| | 0.166 | 0.668 | 0.662 (0.005) | 0.989% | 0.712 | 0.711 (0.006) | 0.215% | 0.753 | 0.756 (0.006) | -0.366% |
| | 0.25 | 1.034 | 1.024 (0.007) | 0.902% | 1.091 | 1.085 (0.007) | 0.482% | 1.143 | 1.144 (0.008) | -0.092% |
| | 0.333 | 1.356 | 1.345 (0.009) | 0.850% | 1.422 | 1.419 (0.009) | 0.206% | 1.482 | 1.486 (0.008) | -0.269% |
| | 0.416 | 1.645 | 1.632 (0.008) | 0.758% | 1.717 | 1.713 (0.010) | 0.229% | 1.782 | 1.788 (0.012) | -0.330% |
| | 0.5 | 1.906 | 1.892 (0.011) | 0.726% | 1.983 | 1.975 (0.010) | 0.392% | 2.052 | 2.055 (0.012) | -0.166% |
| | 0.583 | 2.144 | 2.129 (0.011) | 0.712% | 2.225 | 2.217 (0.013) | 0.357% | 2.297 | 2.297 (0.014) | -0.010% |
| | 0.666 | 2.364 | 2.349 (0.011) | 0.636% | 2.448 | 2.444 (0.010) | 0.184% | 2.522 | 2.527 (0.012) | -0.169% |
| | 0.75 | 2.568 | 2.549 (0.013) | 0.738% | 2.655 | 2.648 (0.015) | 0.284% | 2.731 | 2.732 (0.016) | -0.036% |
| | 0.833 | 2.759 | 2.743 (0.013) | 0.591% | 2.848 | 2.844 (0.015) | 0.162% | 2.925 | 2.934 (0.015) | -0.330% |
| | 0.916 | 2.938 | 2.919 (0.010) | 0.632% | 3.029 | 3.022 (0.011) | 0.240% | 3.107 | 3.117 (0.012) | -0.313% |
| | 1 | 3.106 | 3.090 (0.010) | 0.526% | 3.199 | 3.196 (0.015) | 0.120% | 3.278 | 3.287 (0.015) | -0.260% |
| | 2 | 4.607 | 4.610 (0.019) | -0.075% | 4.721 | 4.741 (0.020) | -0.421% | 4.815 | 4.846 (0.024) | -0.633% |
| | 3 | 5.579 | 5.599 (0.019) | -0.354% | 5.719 | 5.745 (0.017) | -0.445% | 5.838 | 5.865 (0.021) | -0.453% |

Note: std is the standard deviation. The results are shown in percentage.

We illustrate the accuracy of $P_{SV}$ in ITM, ATM and OTM respectively in Figure 3. The setting of the parameters is the same as the initial setting except for the correlation coefficient $\rho$ which is set at 0.5. The time to maturity is from the sixth month to the ninth month. If we set the time to maturity from a month to a year, the figure would be too small for observation. It is seen that all the prices of $P_{SV}$ lie within the 95% confidence interval (C.I). This shows that our analytical solution is accurate.

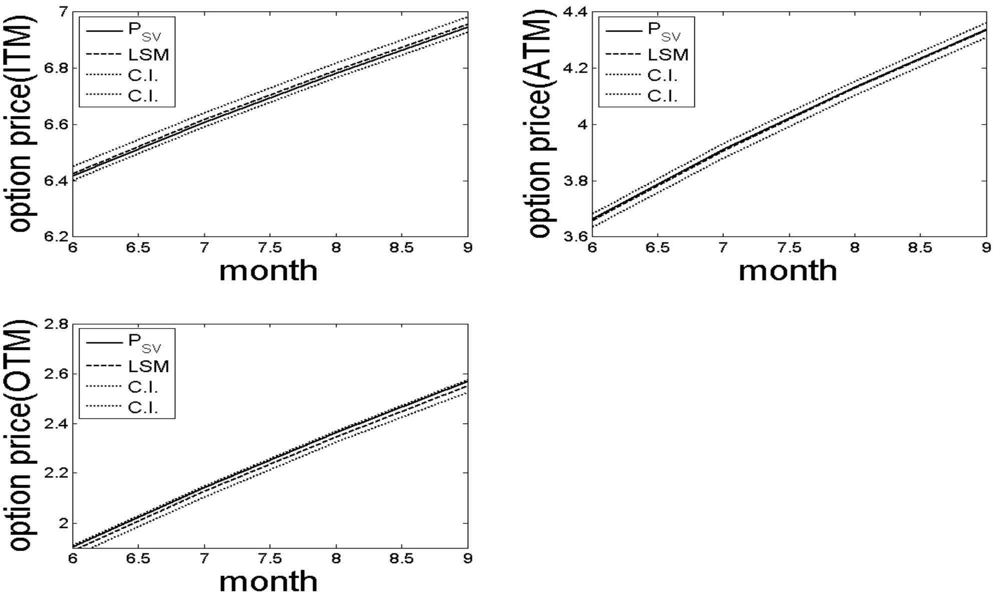


**Figure 3: Accuracy of analytical solution**

In the speed of calculation, the average time taken for $P_{SV}$ is 6.109s while the average time taken for the LSM method varies under different conditions of moneyness. In the aspect of ITM where early exercise is more likely, the average time taken for a six-month option and a one-year option for LSM is about 745.453s and 1415.469s respectively. It is about 598.938s and 1177.109s for ATM. In OTM, it is about 444.859s and 936.031s due to the low possibility of exercising. Compared with the time taken in calculating a six-month option and a one-year option by the LSM method, the efficiency of $P_{SV}$ is 1.22E+02 and 2.32E+02 for ITM, 9.80E+01 and 1.93E+02 for ATM, and 7.28E+01 and 1.53E+02 for OTM. Therefore, it is evident that the speed of the analytical solution is much faster than the LSM method.

Table 4 shows the difference between the stochastic volatility model $P_{SV}$ and the constant volatility model $P_{GJ}$ (Geske and Johnson (1984)). The range of variance $\upsilon$ varies from 20% to 60%. The time to maturity is half a year. The rest of the parameters are the same as the initial setting.

**Table 4 Comparison of difference between $P_{SV}$ and $P_{GJ}$ under different variances**

| $T$ =0.5(yr) | | $\rho$ =0.5 | | | $\rho$ =0 | | | $\rho$ =-0.5 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $S$ | $\nu$ | $P_{SV}$ | $P_{GJ}$ | diff. | $P_{SV}$ | $P_{GJ}$ | diff. | $P_{SV}$ | $P_{GJ}$ | diff. |
| 45 | 0.04 | 5.406 | 5.348 | 0.059 | 5.319 | 5.348 | -0.029 | 5.237 | 5.348 | -0.110 |
| | 0.0625 | 5.876 | 5.812 | 0.064 | 5.789 | 5.812 | -0.023 | 5.702 | 5.812 | -0.110 |
| | 0.09 | 6.421 | 6.348 | 0.073 | 6.337 | 6.348 | -0.011 | 6.253 | 6.348 | -0.096 |
| | 0.1225 | 7.005 | 6.919 | 0.085 | 6.925 | 6.919 | 0.006 | 6.843 | 6.919 | -0.076 |
| | 0.16 | 7.610 | 7.508 | 0.103 | 7.533 | 7.508 | 0.026 | 7.454 | 7.508 | -0.054 |
| | 0.2025 | 8.228 | 8.103 | 0.125 | 8.153 | 8.103 | 0.050 | 8.076 | 8.103 | -0.027 |
| | 0.25 | 8.854 | 8.700 | 0.154 | 8.779 | 8.700 | 0.079 | 8.703 | 8.700 | 0.003 |
| | 0.3025 | 9.484 | 9.296 | 0.188 | 9.409 | 9.296 | 0.113 | 9.334 | 9.296 | 0.038 |
| | 0.36 | 10.116 | 9.887 | 0.229 | 10.041 | 9.887 | 0.154 | 9.966 | 9.887 | 0.079 |
| 50 | 0.04 | 2.252 | 2.313 | -0.061 | 2.272 | 2.313 | -0.041 | 2.284 | 2.313 | -0.029 |
| | 0.0625 | 2.957 | 2.990 | -0.034 | 2.966 | 2.990 | -0.025 | 2.969 | 2.990 | -0.021 |
| | 0.09 | 3.658 | 3.668 | -0.009 | 3.659 | 3.668 | -0.009 | 3.656 | 3.668 | -0.012 |
| | 0.1225 | 4.358 | 4.342 | 0.015 | 4.352 | 4.342 | 0.009 | 4.343 | 4.342 | 0.001 |
| | 0.16 | 5.055 | 5.013 | 0.042 | 5.043 | 5.013 | 0.030 | 5.029 | 5.013 | 0.016 |
| | 0.2025 | 5.750 | 5.677 | 0.073 | 5.733 | 5.677 | 0.055 | 5.714 | 5.677 | 0.037 |
| | 0.25 | 6.443 | 6.335 | 0.108 | 6.421 | 6.335 | 0.085 | 6.398 | 6.335 | 0.062 |
| | 0.3025 | 7.134 | 6.986 | 0.148 | 7.107 | 6.986 | 0.121 | 7.079 | 6.986 | 0.093 |
| | 0.36 | 7.822 | 7.628 | 0.194 | 7.790 | 7.628 | 0.162 | 7.759 | 7.628 | 0.131 |
| 55 | 0.04 | 0.698 | 0.829 | -0.131 | 0.822 | 0.829 | -0.007 | 0.923 | 0.829 | 0.095 |
| | 0.0625 | 1.266 | 1.379 | -0.113 | 1.372 | 1.379 | -0.006 | 1.463 | 1.379 | 0.084 |
| | 0.09 | 1.892 | 1.978 | -0.086 | 1.979 | 1.978 | 0.001 | 2.056 | 1.978 | 0.078 |
| | 0.1225 | 2.548 | 2.604 | -0.056 | 2.619 | 2.604 | 0.014 | 2.682 | 2.604 | 0.078 |
| | 0.16 | 3.222 | 3.246 | -0.024 | 3.278 | 3.246 | 0.032 | 3.330 | 3.246 | 0.084 |
| | 0.2025 | 3.907 | 3.895 | 0.012 | 3.951 | 3.895 | 0.056 | 3.991 | 3.895 | 0.096 |
| | 0.25 | 4.599 | 4.547 | 0.052 | 4.632 | 4.547 | 0.085 | 4.662 | 4.547 | 0.115 |
| | 0.3025 | 5.295 | 5.199 | 0.096 | 5.318 | 5.199 | 0.119 | 5.340 | 5.199 | 0.141 |
| | 0.36 | 5.993 | 5.847 | 0.146 | 6.007 | 5.847 | 0.161 | 6.021 | 5.847 | 0.174 |

The results in Table 4 reveal that as variance $\upsilon$ increases, the difference becomes greater, and they have verified our initial assumption that stochastic volatility model should be more realistic than constant volatility model. Compared with traditional constant volatility model, it is more practical and reasonable to adopt stochastic volatility model.

## 5. Empirical Performance of Analytical Solution Model

To test our analytical solution $P_{SV}$ in the real financial market application, we perform empirical studies. First, we decide on the type of derivatives to be evaluated. Next, we will do an estimation of the parameters. With that, we will verify if stochastic volatility model is superior to constant volatility model in pricing American put options in the actual market.

### 5.1 Resource of Information

Since our analytical solution model $P_{SV}$ evaluates on plain vanilla American put options with stochastic volatility, the empirical data used must comply with such condition. Warrants and options are similar as the two contractual financial instruments are discretionary and have expiration dates. Therefore, we decided to use the put warrants in Taiwan financial market as our empirical study. Thirty put warrants are carefully chosen as samples. The time period selected for the put warrant contracts is from January 2006 to May 2009. Relevant information such as the past market prices and the past stock prices of the underlying assets corresponding to the warrants are collected from the Market Observation Post System (M.O.P.S) and Taiwan Economic Journal (TEJ). We used money market interest rate CP2-90 as the risk-free interest rate. Table 5 provides the information of the thirty put warrants in Taiwan financial market, including their warrant code, underlying stock number, and the time to maturity.

**Table 5 Information of Put Warrant in Taiwan Market**

| Warrant No. | Warrant code | Underlying stock No. | Time to maturity |
|---|---|---|---|
| 05818P | Capital NN | 2002 | 20080924～20090323 |
| 05813P | Capital NJ | 2330 | 20080923～20090323 |
| 05752P | Capital MZ | 2498 | 20080910～20090309 |
| 05722P | Capital MV | 2454 | 20080908～20090309 |
| 05297P | JS J2 | 2498 | 20080722～20090121 |
| 04258P | YT JN | 2618 | 20080502～20081103 |
| 04112P | YTJB | 2834 | 20080421～20081020 |
| 08024P | PSC DB | 0050 | 20080408～20081007 |
| 08022P | YT FP | 0050 | 20080318～20080917 |
| 07366P | YTCQ | 2409 | 20071205～20080704 |
| 06603P | YTAB | 2330 | 20071002～20080505 |
| 06344P | JSB3 | 2498 | 20070907～20080407 |
| 06268P | YCJD | 2449 | 20070828～20080227 |
| 04381P | YCAW | 2448 | 20060921～20070320 |
| 04369P | GCSCB6 | 3474 | 20060915～20070314 |
| 04347P | YCAS | 1504 | 20060908～20070307 |
| 04348P | YCAT | 3019 | 20060908～20070307 |
| 04342P | YCAR | 5534 | 20060907～20070306 |
| 04339P | GCSCB4 | 2501 | 20060905～20070305 |
| 04331P | YCAP | 1722 | 20060831～20070301 |
| 04332P | YCAQ | 2446 | 20060831～20070301 |
| 04319P | YCAL | 2408 | 20060830～20070301 |
| 04318P | GCSCB2 | 3009 | 20060829～20070301 |
| 03734P | YCF3 | 2308 | 20051230～20060629 |
| 03736P | YCF5 | 2393 | 20051230～20060629 |
| 03556P | YCC7 | 3231 | 20051028～20060427 |
| 03554P | YCC5 | 2301 | 20051027～20060426 |
| 03555P | YCC6 | 2331 | 20051027～20060426 |
| 03478P | JS66 | 3009 | 20051013～20060412 |
| 03274P | Masterlink 52 | 1326 | 20050818～20060517 |

## 5.2 Parameter Estimation

To determine if the stochastic volatility model is superior to the constant volatility model, we compared $P_{SV}$ with the constant volatility model $P_{GJ}$ proposed by Geske and Johnson (1984). Both models applied three-point Richardson extrapolation to approach. For $P_{GJ}$, the most important parameter is the estimation of constant volatility. However, for $P_{SV}$, parameters such as the rate of mean reversion $\kappa$, the long term variance $\theta$, volatility of variance $\sigma$, current variance $\upsilon$ and correlation coefficient $\rho$ are all considered crucial.. Therefore, we follow the concept and methods found in Bakshi, Cao and Chen (1997) paper to estimate these parameters.

Let $N$ be the number of the total put warrants. For each $n = 1, \ldots, N$. Let $T_n$ be the time to maturity of the $n$-th put warrant, and $K_n$ is the strike price. Let $\hat{P}_n(T_n ; K_n)$ be the market price (i.e., the observe price). $P_n(T_n ; K_n)$ is the analytical solution price (i.e., the model price). The difference between $\hat{P}_n$ and $P_n$ is a function of the values taken by $\nu$ and by $\Phi = \{\kappa, \theta, \sigma, \rho\}$. For each $n$, define

$$\varepsilon_n[\nu, \Phi] \equiv \hat{P}_n(T_n ; K_n) - P_n(T_n ; K_n) \tag{29}$$

Then, find $\nu$ and parameter vector $\Phi$, to solve

$$SSE \equiv \min_{\nu, \Phi} \sum_{n=1}^{N} \left| \varepsilon_n[\nu, \Phi] \right|^2 \tag{30}$$

iteratively by using the Exhaustive Attack method until a set of parameters is obtained when $SSE$ is at its smallest. The result will hence be the estimated parameters that are needed to be substituted into the models for the calculation of the put warrant prices.

### 5.3 Empirical Performance

We set the put warrant market prices obtained during large volume as the benchmark because the prices are more stable at that time. Then, $P_{SV}$ and $P_{GJ}$ are applied to calculate the prices of the thirty put warrants respectively. Finally, root mean squared error (RMSE) of the prices obtained by each model will be calculated for comparison. The smaller the RMSE, the difference between the model price and the put warrant market price is smaller.

$$RMSE = \sqrt{\frac{\sum_{n=1}^{N} \left( \hat{P}_n(T_n ; K_n) - P_n(T_n ; K_n) \right)^2}{N}} \tag{31}$$

Table 6 shows the market prices of the put warrants, the prices of the put warrants calculated by $P_{SV}$ and $P_{GJ}$ and the respective RMSE obtained. The results indicate that the RMSE of the put warrant prices under $P_{SV}$ is 0.123 while the put warrant prices under $P_{GJ}$ is 0.129. The RMSE obtained by $P_{SV}$ is smaller than that of $P_{GJ}$, proving that stochastic volatility model is superior and efficient for the actual market.

**Table 6 Empirical Performance of the Stochastic Volatility Model**

| Warrant No. | Warrant code | Market price | Constant volatility option price | Stochastic volatility option price |
|---|---|---|---|---|
| 05818P | Capital NN | 2.3 | 2.179 | 2.182 |
| 05813P | Capital NJ | 1.6 | 1.522 | 1.532 |
| 05752P | Capital MZ | 0.46 | 0.473 | 0.483 |
| 05722P | Capital MV | 2.53 | 2.509 | 2.578 |
| 05297P | JS J2 | 0.2 | 0.236 | 0.227 |
| 04258P | YT JN | 2.1 | 2.163 | 2.165 |
| 04112P | YTJB | 0.26 | 0.272 | 0.271 |
| 08024P | PSC DB | 0.24 | 0.242 | 0.244 |
| 08022P | YT FP | 0.95 | 1.119 | 1.114 |
| 07366P | YTCQ | 3.33 | 3.587 | 3.549 |
| 06603P | YTAB | 1.21 | 1.275 | 1.259 |
| 06344P | JSB3 | 0.63 | 0.911 | 0.909 |
| 06268P | YCJD | 0.82 | 0.799 | 0.830 |
| 04381P | YCAW | 1.11 | 1.222 | 1.226 |
| 04369P | GCSCB6 | 0.27 | 0.274 | 0.273 |
| 04347P | YCAS | 0.93 | 0.851 | 0.862 |
| 04348P | YCAT | 1.23 | 1.499 | 1.491 |
| 04342P | YCAR | 1.15 | 1.187 | 1.186 |
| 04339P | GCSCB4 | 0.11 | 0.124 | 0.122 |
| 04331P | YCAP | 0.53 | 0.535 | 0.532 |
| 04332P | YCAQ | 0.85 | 0.826 | 0.834 |
| 04319P | YCAL | 1.47 | 1.612 | 1.607 |
| 04318P | GCSCB2 | 0.17 | 0.175 | 0.172 |
| 03734P | YCF3 | 0.62 | 0.504 | 0.514 |
| 03736P | YCF5 | 0.25 | 0.273 | 0.255 |
| 03556P | YCC7 | 0.51 | 0.906 | 0.905 |
| 03554P | YCC5 | 0.93 | 0.914 | 0.930 |
| 03555P | YCC6 | 0.55 | 0.448 | 0.481 |
| 03478P | JS66 | 0.13 | 0.104 | 0.115 |
| 03274P | Masterlink 52 | 0.17 | 0.173 | 0.172 |
| Root Mean Squared Error (RMSE) | | | 0.129 | 0.123 |

## 6. Conclusion

Much of the literature on American options is evaluated under constant volatility. In this paper, we consider the harder problem of deriving an analytical solution by using barrier options to evaluate American put options with stochastic volatility. Our model proves to be accurate and efficient with relative error less than 1% in numerical analyses. From the empirical result, it is shown that stochastic volatility model indeed performs better than traditional constant volatility model in evaluating financial derivatives. In other words, stochastic volatility model can better illustrate the real market. Our analytical solution is practical because it can be applied broadly on any options that satisfy the conditions of plain vanilla American put.

Besides pricing options, hedge ratio (Greeks) is another section that can be further explored. Theoretically, if differentiation is performed on our model, other analytical solutions for hedge ratio could also be derived. But still, we strive to develop the closed-form for American options with stochastic volatility in future studies.

In addition, Black and Scholes (1973) model has treated shareholders' equity as a standard call option with corporate value as the underlying asset. Black and Scholes (1973) assumed the call option to be path-independent. Instead of being influenced by the trend of the asset value within the duration, the profit and loss (P&L) of the option will only be enhanced on the expiration date. Moreover, the equity value will become nothing if the corporate declares bankrupt when the corporate value is less than the liability during the monitoring period. To modify such unsuitable assumption, the equity value can be treated as a down-and-out barrier call option with corporate value as the underlying asset, while the bond value as the strike price. With that, the characteristic of path-dependent can then be captured. This is also another area we can investigate since we have already derived a discrete stochastic volatility barrier option model in this paper. We suggest that future research can be done by using the barrier options on the evaluation of the credit risks of a corporate.

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